Topology and its Applications 25 (1987) 113-120
North-Holland



# ON THE GENUS OF GROUPS WITH OPERATORS

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Received 26 July 1985

Charles Cassidy has given a definition of genus for nilpotent groups with operators, generalizing the Mislin genus. Here we modify Cassidy's definition to bring it into line with homotopy theory. Thus, with our definition, if $X$ and $Y$ are nilpotent spaces in the same genus, then the $\pi_1 X$-group $\pi_n X$ is in the genus of the $\pi_1 Y$-group $\pi_n Y$. Relations are found between the genus of the $Q$-group $N$ and the (Mislin) genus of the semidirect product $N \rtimes Q$.

AMS (MOS) Subj. Class.: 20E99, 20J05

groups, operators, semidirect product, homotopy theory, genus

## Introduction

In [5] (see also [4]) Mislin introduced the notion of the *genus* of a finitely generated nilpotent group $N$. Thus $G(N)$ consists of isomorphism classes of finitely generated nilpotent groups $M$ such that, at every prime $p$, the localizations $M_p$ and $N_p$ are isomorphic. If, in particular, $N$ has finite commutatorsubgroup, then $G(N)$ may be given the structure of a finite abelian group with $N$ as zero element; this was shown in [5] and further results were obtained in [4].

In [1] Cassidy generalized the notion of genus from nilpotent groups to nilpotent $Q$-groups. That is, we fix a group $Q$ and we suppose that $Q$ acts on the finitely generated nilpotent group $N$. The group $Q$ then also acts on the localizations, $N_P$, of $N$, so that there is available an evident generalization of the notion of genus. Cassidy studied this generalization, obtaining extensions of theorems of Mislin, and in particular, he produced examples of non-trivial *Q-genus* related in a natural way to Mislin's examples in [5]. Precisely he constructed nilpotent $Q$-groups $N(p, m)$ depending on an odd prime $p$ and an integer $m$ satisfying $1 \leq m \leq \frac{1}{2}(p-1)$, such that $N(p, m)$, $N(p, n)$ are not isomorphic but are evidently in the same $Q$-genus; here $Q = C^{(p-1)/2}$. We then recover Mislin's examples by forming the semidirect product $N(p, m) \rtimes Q$.

Suppose that $Q$ acts nilpotently on $N$ in the sense of [2]; thus we form the *lower central Q-series*

$$\Gamma_Q^0 N = N, \quad \Gamma_Q^{i+1} N = gp\langle a \cdot xb \cdot a^{-1} b^{-1} \rangle, \quad a \in N, \quad b \in \Gamma_Q^i N, \quad x \in Q, \quad i \geqslant 0, \tag{0.1}$$

and we suppose that $\Gamma_Q^c N = \{1\}$ for some $c$. As pointed out by Cassidy, under these conditions $Q_p$ acts on $N_p$ in a way compatible with the action of $Q$ on $N$, but we do not change the notion of $Q$-genus by asking that $M_p$ be isomorphic to $N_p$ as $Q_p$-groups rather than as $Q$-groups.

However, although the Cassidy definition of genus is well adapted to the purpose of establishing close relation to the Mislin genus, it is not really what one wants when one is considering the relationship of the genus in homotopy theory to the Mislin genus in (nilpotent) group theory. Consider two nilpotent spaces, of finite type, $X$ and $Y$, in the same genus. Then $\pi_1 X$, $\pi_1 Y$ are two nilpotent groups in the same genus, which we may call $Q$, $R$ respectively. Since, for any $n \geqslant 2$, $\pi_n X \cong \pi_n Y$, let us designate either group by the common letter $A$. Then $A$ admits the structure of a nilpotent $Q$-module and nilpotent $R$-module. Moreover, for each prime $p$, there is a homotopy equivalence $X_p \cong Y_p$ which gives rise to an isomorphism $\theta : Q_p \to R_p$ and an automorphism $\alpha : A_p \to A_p$ of the abelian group structure such that

$$\alpha(xa) = (\theta x)(\alpha a), \qquad x \in Q_p, \quad a \in A_p. \tag{0.2}$$

This suggests a broader, weaker notion of genus. We consider the category of groups with operators, $\mathscr{G}_{\mathrm{op}}$. Thus an object of $\mathscr{G}_{\mathrm{op}}$ is a pair $(\pi, G)$ consisting of a group $\pi$ and a $\pi$-group $G$; and a morphism of $\mathscr{G}_{\mathrm{op}}$ is a pair $(f, \alpha) : (\pi, G) \to (\bar{\pi}, \bar{G})$ where $f : \pi \to \bar{\pi}$ is a group-homomorphism and $\alpha : G \to \bar{G}$ is a group-homomorphism satisfying

$$\alpha(xg) = (fx)(\alpha g), \qquad x \in \pi, \quad g \in G. \tag{0.3}$$

If we confine attention to pairs $(\pi, G)$ where $\pi$ is nilpotent and acts nilpotently on $G$, we obtain a full subcategory $N\mathscr{G}_{\mathrm{op}}$ of $\mathscr{G}_{\mathrm{op}}$. In this subcategory $P$-localization, where $P$ is an arbitrary family of primes, makes good sense (see [2] or Section 1 of this paper). Thus if we further restrict to the case where both $\pi$ and $G$ are finitely generated as groups, we get a notion of genus which evidently has the following two very satisfactory properties.

**Theorem 0.1.** *Let $X$, $Y$ be nilpotent spaces of finite type whose homotopy types are in the same genus. Then, for any $n \geqslant 2$, the isomorphism classes in $N\mathscr{G}_{\mathrm{op}}$ of $(\pi_1 X, \pi_n X)$ and $(\pi_1 Y, \pi_n Y)$ are in the same genus.*

**Theorem 0.2.** *Let $(\pi, G)$, $(\bar{\pi}, \bar{G})$ be objects whose isomorphism classes in $N\mathscr{G}_{\mathrm{op}}$ are in the same genus. Then the isomorphism classes of the semidirect products $G \rtimes \pi$ and $\bar{G} \rtimes \bar{\pi}$ are in the same genus.*

We remark that, of course, if $Q$ acts nilpotently on the groups $N$ and $\bar{N}$, and if $N$ and $\bar{N}$ are then in the same Cassidy genus, then certainly $(Q, N)$ and $(Q, \bar{N})$ are in the same genus in $N\mathcal{G}_{\mathrm{op}}$. However, the converse is obviously false. For if $A$ and $\bar{A}$ are $C$-modules whose underlying abelian group is $Z/9$ with $C$-actions given by

$$\eta \cdot a = 4a, \qquad \eta \cdot \bar{a} = 7\bar{a}, \quad \text{where } C = \langle \eta \rangle, \quad a \in A, \quad \bar{a} \in \bar{A},$$

then it is easy to see that $A$ and $\bar{A}$ are nilpotent $C$-modules which are not in the same Cassidy genus but which are actually isomorphic in $N\mathcal{G}_{\mathrm{op}}$.

One of our main purposes in Section 1 is to obtain a converse of Theorem 0.2 under certain restrictive hypotheses. These hypotheses consist of restricting the pairs $(Q, N)$ in $N\mathcal{G}_{\mathrm{op}}$ by requiring that $Q$ be torsionfree and $N$ finite commutative. This restriction fortunately does not prevent us from applying our result to the examples of Mislin genus which were fully described in [3]. We are thus able to make a complete calculation of the genus $G(Q, N)$ in certain interesting, non-trivial cases. These calculations are carried out in Section 2.

## 1. The genus of groups with operators

Let $Q$ be a nilpotent group, which will be fixed throughout this preliminary discussion. We consider the category of $Q$-groups $N$, and we restrict attention to *nilpotent* $Q$-actions; recall that in this case $N$ is necessarily nilpotent as a group [2]. Indeed, we have more, namely,

**Proposition 1.1.** *Let $N$ be a $Q$-group. Then the action is nilpotent if and only if the semidirect product $N \rtimes Q$ is nilpotent.*

Let us write $\mathrm{Nact}_Q\, N$ for the set of nilpotent actions of $Q$ on $N$. Thus an element of $\mathrm{Nact}_Q\, N$ is a homomorphism $w : Q \to \mathrm{Aut}\, N$.

**Theorem 1.2.** *Let $P$ be a family of primes. For a given $w \in \mathrm{Nact}_Q\, N$, there exists a unique $w_P \in \mathrm{Nact}_{Q_P}\, N_P$ such that the diagram*

$$\begin{array}{ccc} Q & \xrightarrow{w} & \mathrm{Aut}\, N \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e_*} \\ Q_P & \xrightarrow{w_P} & \mathrm{Aut}\, N_P \end{array} \tag{1.1}$$

*commutes.*

**Proof** [2]. Obviously $e_*$ defines a function

$$e_\dagger\colon\ \mathrm{Nact}_Q\, N \to \mathrm{Nact}_Q N_P.$$

Thus our Theorem is proved when we show that

$$e\colon\ Q \to Q_P \text{ sets up a bijection}$$

$$e^*\colon\ \mathrm{Nact}_{Q_P}\, N_P \to \mathrm{Nact}_Q\, N_P. \tag{1.2}$$

We construct an inverse to $e^*$ as follows. Let $\eta \in \mathrm{Nact}_Q N_P$ give rise to the semidirect product

$$\begin{tikzcd} N_P \arrow[r, tail, "i"'] & E \arrow[r, two heads, shift right, "k"'] & Q \arrow[l, shift right, "s"'] \end{tikzcd} \tag{1.3}$$

with canonical cross-section $s$. We $P$-localize (1.3) to obtain the commutative diagram

$$\begin{tikzcd} \eta\colon N_P \arrow[r, tail, "i"'] \arrow[d, equal] & E \arrow[r, two heads, shift right, "k"'] \arrow[d, "e"'] & Q \arrow[l, shift right, "s"'] \arrow[d, "e"'] \\ \eta_P\colon N_P \arrow[r, tail, "i_P"'] & E_P \arrow[r, two heads, shift right, "k_P"'] & Q_P \arrow[l, shift right, "s_P"'] \end{tikzcd} \tag{1.4}$$

Then the bottom extension in (1.4) determines an action of $Q_P$ on $N_P$ which is nilpotent because $E_P$ is nilpotent. Thus $\eta \to \eta_P$ determines a function $L = L_P$: $\mathrm{Nact}_Q N_P \to \mathrm{Nact}_{Q_P} N_P$. Moreover, it is easy to see from (1.4) that

$$x \cdot a = ex \cdot a, \qquad a \in N_P, \quad x \in Q,$$

so that $\eta = e^*(\eta_P) = e^* L(\eta)$.

Now consider an artibrary $\bar{\eta} \in \mathrm{Nact}_{Q_P} N_P$, represented as

$$\begin{tikzcd} \bar{\eta}\colon N_P \arrow[r, tail, "\bar{i}"'] & \bar{E} \arrow[r, two heads, shift right, "\bar{k}"'] & Q_P \arrow[l, shift right, "\bar{s}"'] \end{tikzcd}$$

and pull back by means of $e: Q \to Q_P$; we obtain, say,

$$\begin{tikzcd} \eta\colon N_P \arrow[r, tail, "i"'] \arrow[d, equal] & E \arrow[r, two heads, shift right, "k"'] \arrow[d, "f"] & Q \arrow[l, shift right, "s"'] \arrow[d, "e"] \\ \bar{\eta}\colon N_P \arrow[r, tail, "\bar{i}"'] & \bar{E} \arrow[r, two heads, shift right, "\bar{k}"'] & Q_P \arrow[l, shift right, "\bar{s}"'] \end{tikzcd}$$

Then $\eta = e^*(\bar{\eta})$ and $f: E \to \bar{E}$ $P$-localizes. Thus $\bar{\eta} = L(\eta) = Le^*(\bar{\eta})$ and $L$ is, as claimed, inverse to $e^*$. $\square$

**Remark.** Obviously $e^*$ extends to a function $\mathrm{Act}_{Q_P} N_P \to \mathrm{Act}_Q N_P$ (where we do not restrict attention to nilpotent actions). We do not believe that the function $L$ extends similarly.

We consider now the category $\mathcal{G}_{\mathrm{op}}$ of operator groups $(Q, N)$ described in the Introduction, and the full subcategory $N\mathcal{G}_{\mathrm{op}}$ consisting of pairs $(Q, N)$ where $Q$ is nilpotent and $N$ is $Q$-nilpotent. Thus the morphisms are pairs $(f, \alpha)$: $(Q_1, N_1) \to (Q_2, N_2)$ consisting of homomorphisms $f$: $Q_1 \to Q_2$, $\alpha$: $N_1 \to N_2$ satisfying

$$\alpha(x \cdot a) = fx \cdot \alpha a, \qquad x \in Q_1, \quad a \in N_1. \tag{1.5}$$

Note that we then have a diagram

$$\begin{tikzcd} N_1 \arrow[r, tail] \arrow[d, "\alpha"] & E_1 \arrow[r, two heads, shift right] \arrow[d, "g"] & Q_1 \arrow[l, shift right] \arrow[d, "f"] \\ N_2 \arrow[r, tail] & E_2 \arrow[r, two heads, shift right] & Q_2 \arrow[l, shift right] \end{tikzcd} \tag{1.6}$$

where $g$ is given by $g(a, x) = (\alpha a, fx)$, $a \in N_1$, $x \in Q_1$.

In the case in which $(f, \alpha) \in N\mathscr{G}_{\mathrm{op}}$, we may localize (1.6) at the family of primes $P$ and it is then seen that, as in Theorem 1.2, we obtain $Q_{iP}$-nilpotent $Q_{iP}$-groups $N_{iP}$, $i = 1, 2$, and $\alpha : N_1 \to N_2$ $P$-localizes to $\alpha_P : N_{1P} \to N_{2P}$, satisfying

$$\alpha_P(x_P, a_P) = f_P x_P \cdot \alpha_P a_P, \qquad x_P \in Q_{1P}, \quad a_P \in N_{1P}, \tag{1.7}$$

where $f_P : Q_{1P} \to Q_{2P}$ $P$-localizes $f : Q_1 \to Q_2$.

We are thus ready to define the genus of a $Q$-nilpotent $Q$-group $N$, in the case that $Q$ and $N$ are finitely generated as groups. Write $FN\mathscr{G}_{\mathrm{op}}$ for the full subcategory consisting of such pairs $(Q, N)$.

**Definition 1.1.** Let $(Q, N)$ be in $FN\mathscr{G}_{\mathrm{op}}$. Then the *genus* of $(Q, N)$ is the family of isomorphism classes $\{R, M\}$ of objects of $FN\mathscr{G}_{\mathrm{op}}$ such that $(Q_p, N_p) \cong (R_p, M_p)$ in $N\mathscr{G}_{\mathrm{op}}$ for all primes $p$.

We write $G(Q, N)$ for the genus of $(Q, N)$. Observe that, if $(Q_1, N_1) \in G(Q, N)$, then $Q_1 \in G(Q)$, $N_1 \in G(N)$ in the sense of Mislin. The formation of the semidirect product obviously induces a function from $G(Q, N)$ to $G(N \rtimes Q)$, where $G(\ )$ is the Mislin genus [4, 5]. We write this function

$$S:\ G(Q, N) \to G(N \rtimes Q). \tag{1.8}$$

We observe the following result which will be applied in the next section.

**Theorem 1.3.** *If $Q$ is torsion-free and $N$ is finite commutative, then $S : G(Q, N) \to G(N \rtimes Q)$ is bijective.*

**Proof.** If $S(Q_1, N_1) = S(Q_2, N_2)$, then $N_1 \rtimes Q_1 \cong N_2 \rtimes Q_2$. Since $N$ is finite and $Q$ is torsionfree, it follows that $N_1 \cong N_2 \cong N$ and $Q_1$, $Q_2$ are torsionfree. Thus $N_1$, $N_2$ are the torsionsubgroups of $N_1 \rtimes Q_1$, $N_2 \rtimes Q_2$ respectively, so that the isomorphism $N_1 \rtimes Q_1 \cong N_2 \rtimes Q_2$ yields the diagram

$$\begin{tikzcd} N_1 \arrow[r] \arrow[d, "\alpha"', "\cong"] & N_1 \rtimes Q_1 \arrow[r, two heads] \arrow[d, "\beta"', "\cong"] & Q_1 \arrow[d, "\gamma"', "\cong"] \\ N_2 \arrow[r] & N_2 \rtimes Q_2 \arrow[r, two heads] & Q_2 \end{tikzcd} \tag{1.9}$$

Since $N_1$, $N_2$ are commutative, we infer from (1.9) that

$$\alpha(x \cdot a) = \gamma x \cdot \alpha a, \qquad a \in N_1, \quad x \in Q_1,$$

so that $(Q_1, N_1) \cong (Q_2, N_2)$ in $N\mathscr{G}_{\mathrm{op}}$. This shows that $S$ is injective.

Let us now suppose that $B \in G(N \rtimes Q)$. Let $T$ be the torsion subgroup of $B$, with quotient $R$,

$$T \rightarrowtail B \twoheadrightarrow R. \tag{1.10}$$

Since $B_p \cong (N \rtimes Q)_p = N_p \rtimes Q_p$, and since $N_p$ is the torsion subgroup of $N_p \rtimes Q_p$, we infer a diagram, for each prime $p$,

$$\begin{tikzcd} T_p \arrow[r, tail] \arrow[d, "\cong"'] & B_p \arrow[r, two heads] \arrow[d, "\cong"'] & R_p \arrow[d, "\cong"'] \\ N_p \arrow[r, tail] & E_p \arrow[r, two heads] & Q_p, \quad E = N \rtimes Q. \end{tikzcd} \tag{1.11}$$

Let (1.10) represent $\eta \in H^2(R;\, T)$. This makes sense since it follows from (1.11) that $T \cong N$, as (commutative) group. It follows from Proposition 1.4 below that the $p$-localization of $H^2(R;\, T)$ is $H^2(R_p;\, T_p)$. Thus (1.11) tells us that $\eta$ is an element of $H^2(R;\, T)$ all of whose $p$-localizations are zero. We conclude that $\eta = 0$ so that (1.10) splits. Thus (1.10) is a right-split extension with $T \cong N$; and we conclude that $B = T \rtimes R$ for some nilpotent action of $R$ on $T$. Then (1.11) again tells us that $(R, T) \in G(Q, N)$, when $T$ is regarded as an $R$-group via (1.10). Obviously $S(R, T) = B$.

It thus remains only to establish the following proposition.

**Proposition 1.4.** *Let $R$ be a finitely generated nilpotent group and $T$ an $R$-nilpotent $R$-module. Then, for any family of primes $P$ and any $n \geq 1$,*

$$H^n(R;\, T)_P = H^n(R;\, T_P) = H^n(R_P;\, T_P). \tag{1.12}$$

**Proof.** Assume first that the $R$-action on $T$ is trivial. We then have the diagram, for any $n \geq 1$,

$$\begin{tikzcd} \mathrm{Ext}(H_{n-1}R, T) \arrow[r, tail] \arrow[d, "e'_*"] & H^n(R;\, T) \arrow[r, two heads] \arrow[d, "e_*"] & \mathrm{Hom}(H_nR, T) \arrow[d, "e''_*"] \\ \mathrm{Ext}(H_{n-1}R, T_P) \arrow[r, tail] & H^n(R;\, T_P) \arrow[r, two heads] & \mathrm{Hom}(H_nR, T_P) \\ \mathrm{Ext}(H_{n-1}R_P, T_P) \arrow[r, tail] \arrow[u, "e'^*"] & H^n(R_P;\, T_P) \arrow[r, two heads] \arrow[u, "e^*"] & \mathrm{Hom}(H_nR_P, T_P) \arrow[u, "e''^*"] \end{tikzcd} \tag{1.13}$$

Since $R$ is finitely generated, nilpotent it follows that $H_nR$ is finitely generated. Further we know that $H_nR_P = (H_nR)_P$. Thus $e'_*$, $e''_*$ $P$-localize and $e'^*$, $e''^*$ are isomorphisms. Thus the conclusion (1.12) follows from (1.13) in this case.

We now proceed by induction on $\mathrm{nil}_R\, T$. If $\mathrm{nil}_R\, T = c$, form the short exact sequence of $R$-modules

$$\Gamma \rightarrowtail T \twoheadrightarrow U, \tag{1.14}$$

where $\Gamma = \Gamma_R^{c-1} T$ and $U = T/\Gamma$. Then $R$ operates trivially on $\Gamma$ and $\mathrm{nil}_R\, U = c-1$. The sequence (1.14) gives rise to the diagram of coefficient sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(R;U) \arrow[r] \arrow[d, "e_{1*}"] & H^n(R;\Gamma) \arrow[r] \arrow[d, "e_{2*}"] & H^n(R;T) \arrow[r] \arrow[d, "e_{3*}"] & H^n(R;U) \arrow[r] \arrow[d, "e_{4*}"] & H^{n+1}(R;\Gamma) \arrow[r] \arrow[d, "e_{5*}"] & \cdots \\
\cdots \arrow[r] & H^{n-1}(R;U_P) \arrow[r] & H^n(R;\Gamma_P) \arrow[r] & H^n(R;T_P) \arrow[r] & H^n(R;U_P) \arrow[r] & H^{n+1}(R;\Gamma_P) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{n-1}(R_P;U_P) \arrow[r] \arrow[u, "e_1^*"] & H^n(R_P;\Gamma_P) \arrow[r] \arrow[u, "e_2^*"] & H^n(R_P;T_P) \arrow[r] \arrow[u, "e_3^*"] & H^n(R_P;U_P) \arrow[r] \arrow[u, "e_4^*"] & H^{n+1}(R_P;\Gamma_P) \arrow[r] \arrow[u, "e_5^*"] & \cdots
\end{tikzcd}$$

where we know, from our inductive hypothesis, that $e_{1*}$, $e_{2*}$, $e_{4*}$, $e_{5*}$ $P$-localize and $e_1^*$, $e_2^*$, $e_4^*$, $e_5^*$ are isomorphisms. We conclude that $e_{3*}$ $P$-localizes and $e_3^*$ is an isomorphism. This completes the inductive step; with it both Proposition 1.4 and, hence, Theorem 1.3 are proved. □

## 2. The Escher staircase

We recall the example which was central to [3]. We choose a prime $p$ and integers $n$, $k \geqslant 1$. We exclude the following cases: $p=2$, $k=1$ (to avoid falsehood); $p=2$, $n=1,2$; $p=3$, $n=1$ (to avoid triviality). We choose[1] $u = 1+cp^k$, $p \nmid c$ and $l$ semi-primitive modulo $p^n$; that is,

$$\min\{\sigma \mid l^\sigma \equiv \pm 1 \bmod p^n\} = s = \tfrac{1}{2}p^{n-1}(p-1).$$

Finally, let $m$ be such that $lm \equiv 1 \bmod p^n$. We then define nilpotent groups $\bar{N}_0$, $\bar{N}_1, \ldots, \bar{N}_{s-1}$ by the following rule:

$$\bar{N}_i = \langle x, y;\, x^{p^{n+k}} = 1,\, yxy^{-1} = x^{u^{m^i}}\rangle, \quad i = 0, 1, \ldots, s-1.$$

We proved in [3] the following theorems.

**Theorem 2.1.** *The rule $x \mapsto x$, $y \mapsto y^l$ determines a normal embedding of $\bar{N}_i$ in $\bar{N}_{i+1}$ $(i = 0, 1, \ldots, s-1;\ \bar{N}_s = \bar{N}_0)$, with quotient cyclic of order $l$, that is, $C_l$.*

**Theorem 2.2.** *If $\bar{N} = \bar{N}_0$, then $G(\bar{N})$ is a cyclic group of order $s$ with generator $\bar{N}_1$; indeed,*

$$\bar{N}_i = i\bar{N}_1 \quad \textit{in } G(\bar{N}).$$

[1] If $p = 2$, we may take $u = -1 + cp^k$.

**Theorem 2.3.** *If* $\sum_{\lambda=1}^{t} i_\lambda \equiv \sum_{\lambda=1}^{t} j_\lambda \bmod s$, *then* $\prod_{\lambda=1}^{t} \bar{N}_{i_\lambda} \cong \prod_{\lambda=1}^{t} \bar{N}_{j_\lambda}$, *where* $\prod$ *is the direct product.*

It is Theorem 2.1 which leads us to think of the cycle of groups $\bar{N}_i$ and normal embeddings $\bar{N}_i \lhd \bar{N}_{i+1}$ as an 'Escher staircase'.

Now we pointed out in [3] that $\bar{N}_i$ may be regarded as the semidirect product $A_i \rfloor C$, where the underlying abelian group of $A_i$ is $Z/_{p}{}^{n+k}$, and $C = \langle \xi \rangle$ acts on $A_i$ by

$$\xi \cdot a = u^{m^i} a, \quad a \in A_i. \tag{2.1}$$

Then Theorem 2.1 translates into

**Theorem 2.1$_*$.** *There is a morphism (in $\mathcal{G}_{\mathrm{op}}$) from $A_i$ to $A_{i+1}$,*

$$(f, \alpha)\colon\ A_i \to A_{i+1},$$

given by $f\xi = \xi^l$, $\alpha a = a$, $i = 0, 1, \ldots, s-1$ $(A_s = A_0)$.

Notice that this morphism is an isomorphism (indeed, the identity) of the underlying abelian group structures of $A_i$ and $A_{i+1}$, but is not an isomorphism in $\mathcal{G}_{\mathrm{op}}$. Indeed there can of course, be no isomorphism in $\mathcal{G}_{\mathrm{op}}$ (even less in the usual category of $C$-modules) since, if there were, $\bar{N}_i$ and $\bar{N}_{i+1}$ would be isomorphic as groups.

Using Theorem 1.3, we may translate Theorem 2.2 as follows:

**Theorem 2.2$_*$.** *If $A = A_0$, then $G(C, A)$ is a cyclic group of order $s$ with generator $(C, A_1)$; indeed,*

$$(C, A_i) = i(C, A_1) \quad \text{in } G(C, A).$$

Again using Theorem 1.3, we translate Theorem 2.3 as follows:

**Theorem 2.3$_*$.** *If* $\sum_{\lambda=1}^{t} i_\lambda \equiv \sum_{\lambda=1}^{t} j_\lambda \bmod s$, *then* $(C^t, \bigoplus_{\lambda=1}^{t} A_{i_\lambda}) \cong (C^t, \bigoplus_{\lambda=1}^{t} A_{j_\lambda})$ *in* $\mathcal{G}_{\mathrm{op}}$, *where $C^t$ acts on* $\bigoplus_{\lambda=1}^{t} A_{i_\lambda}$, $\bigoplus_{\lambda=1}^{t} A_{j_\lambda}$ *in the obvious way.*

We remark that in the examples discussed in this section, the operator group does not change within the genus, since it is a (finitely generated) commutative group.

## References

[1] C. Cassidy, Le genre d'un groupe nilpotent avec operateurs, Comm. Math. Helv. 53 (1978) 364–384.

[2] P. Hilton, Nilpotent actions on nilpotent groups, Lecture Notes in Mathematics 450 (Springer, Berlin, 1975) 174–196.

[3] P. Hilton, On the genus of nilpotent groups and spaces, Israel J. Math. 54 (1986) 1–13.

[4] P. Hilton and G. Mislin, On the genus of a nilpotent group with finite commutator subgroup, Math. Zeit. 146 (1976) 201–211.

[5] G. Mislin, Nilpotent groups with finite commutator subgroups, Lecture Notes in Mathematics 418 (Springer, Berlin, 1974) 103–118.